WERKCOLLEGE: Voortgezette wiskunde Datum: 070982 

10-3 $\partial_1\partial_2 f(x,y) = \partial_1\partial_2 (y+1)^{xy} = \partial_1 (y+1)^{(x-1)y} x =$

$\partial_1((y+1)x^x + (y+1)x^y) = \partial_1(y+1)x^x + \partial_1(y+1)x^y = (y+1)\partial_1 x^x + (y+1)\partial_1 x^y =$

$(y+1)(e^{x\ln x}(1+\ln x) + x^{y+1})$

$\partial\partial_2 e^{xy\ln(y+1)} = \partial_1 x\cdot e^{xy\ln(y+1)}\left(\ln(y+1) + \frac{y}{y+1}\right)$

$\left(e^{xy\ln(y+1)} + x\cdot e^{xy\ln(y+1)}\cdot y\ln(y+1)\right)\left(\ln(y+1) + \frac{y}{y+1}\right) =$

$(1+xy\ln(y+1))(y+1)^{xy}\left(\ln(y+1) + \frac{y}{y+1}\right)$

10-5 $\partial_1\partial_1 u + \partial_2\partial_2 u = \partial_1(\partial_1 u) + \partial_2(\partial_2 u) = \partial_1(\partial_2 v) + \partial_2(-\partial_1 v) =$

$\partial_1\partial_2 v - \partial_2\partial_1 v = 0$ volgens stelling geldt $\partial_1\partial_2 v = \partial_2\partial_1 v$ als $v$ cont. diff b.

$\partial_1\partial_1 v + \partial_2\partial_2 v = \partial_1(\partial_1 v) + \partial_2(\partial_2 v) = \partial_1(-\partial_2 u) + \partial_2(\partial_1 u) =$

$-\partial_1\partial u + \partial_2\partial_1 u = 0$ volgens stelling geldt $\partial_1\partial_2 v = \partial_2\partial_1 v$ als $v$ cont. diff b.

10-7 $x(u,v) = \frac{u^2}{v}$ en $y(u,v) = \frac{v^2}{u}$ $f(x,y) = \sin xy$

$z(u,v) = f(x(u,v), y(u,v))$

$\frac{\partial z}{\partial v} = \frac{\partial f(x(u,v), y(u,v))}{\partial v} =$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = y\cos(xy)\frac{\partial x}{\partial v} + x\cos(xy)\frac{\partial y}{\partial v} = -y\cos(xy)\frac{u^2}{v^2} + x\cos(xy)\frac{2v}{u}$

~~$\frac{\partial^2 z}{\partial u\partial v} = \frac{\partial z}{\partial v}\left(x\cos(xy)\frac{dy}{dv} + y\cos(xy)\frac{dy}{dv}\right) =$~~ kettingregel

$+ x\cos xy\frac{d}{du}\left(\frac{2v}{u}\right)$

$\frac{\partial^2 z}{\partial u\partial v} = \frac{\partial}{\partial u}(y\cos xy)\cdot\frac{-u^2}{v^2} + y\cos xy\frac{d}{du}\left(\frac{-u^2}{v^2}\right) + \frac{\partial}{\partial u}(x\cos xy)\cdot\frac{2v}{u}$

voorbeeld $\frac{d}{du}(y\cos xy) = \frac{\partial}{\partial x}(y\cos xy)\frac{dx}{du} + \frac{\partial}{\partial y}(y\cos xy)\frac{dy}{du}$

$= -y^2\sin xy\frac{2u}{v} + (\cos xy - xy\sin xy) - \frac{u^2}{u^2}$

ook $\frac{d}{du}(x\cos xy)$ mbv kettingregel uitrekenen!

10-8 $x = r\cos\phi$ $y = r\sin\phi$ $z(r,\phi) = v(x(r,\phi), y(r,\phi))$

$\frac{\partial z}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\phi + \frac{\partial v}{\partial y}\sin\phi$

$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial v}{\partial x}\frac{\partial}{\partial r}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial y}\right)\frac{\partial y}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial}{\partial r}\left(\frac{\partial y}{\partial r}\right)$

voorbeeld $\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 v}{\partial x^2}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 v}{\partial y\partial x}\frac{\partial y}{\partial r}$

ook $\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x\partial y}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 v}{\partial y^2}\cdot\frac{\partial y}{\partial r}$

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$\frac{\partial^2 z}{\partial r^2} = \left(\frac{\partial^2 v}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 v}{\partial y\partial x}\frac{\partial y}{\partial r}\right)\frac{\partial x}{\partial r} + \frac{\partial v}{\partial x}\frac{\partial^2 x}{\partial r^2} + \left(\frac{\partial^2 v}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 v}{\partial y^2}\frac{\partial y}{\partial r}\right)\frac{\partial y}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial^2 y}{\partial r^2}$

$= \frac{\partial^2 v}{\partial x^2}\cos^2\phi + 2\frac{\partial^2 v}{\partial y\partial x}\sin\phi\cos\phi + \frac{\partial^2 v}{\partial y^2}\sin^2\phi$

invullen voor $r, \varphi$

$$\frac{\partial^2 z}{\partial \varphi^2} = \left(\frac{\partial^2 v}{\partial x^2}\frac{\partial x}{\partial \varphi} + \frac{\partial^2 v}{\partial y \partial x}\frac{\partial x}{\partial \varphi}\right)\frac{dx}{d\varphi} + \frac{\partial v}{\partial x}\frac{\partial^2 x}{\partial \varphi^2} + \left(\frac{\partial^2 v}{\partial y^2}\frac{\partial x}{\partial \varphi} + \frac{\partial^2 v}{\partial y \partial x}\frac{\partial y}{\partial \varphi}\right)\frac{dy}{d\varphi} + \frac{\partial v}{\partial y}\frac{\partial^2 x}{\partial \varphi^2}$$

$$= \frac{\partial^2 v}{\partial x^2} r^2 \sin\varphi - 2\frac{\partial^2 v}{\partial x \partial x} r^2 \sin\varphi\cos\varphi + \frac{\partial v}{\partial x}\cdot -r\cos\varphi + \frac{\partial^2 v}{\partial y^2} r\cos^2\varphi + \frac{\partial v}{\partial y}\cdot -r\sin\varphi$$

uitgeschreven geeft $\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \varphi^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$

10-9

$$\frac{\partial H}{\partial x} = \frac{d}{dx} H(x,y) = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

$$\frac{\partial^2 H}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right) =$$

$$\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial x}{\partial x} + \frac{\partial^2 F}{\partial y \partial x}\frac{\partial y}{\partial x} + \frac{\partial^2 F}{\partial x \partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right)\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial F}{\partial z} =$$

$$\left(\frac{\partial^2 F}{\partial x} + \frac{\partial^2 F}{\partial x \partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial x}\left(\frac{\partial^2 F}{\partial z \partial x} + \frac{\partial^2 F}{\partial z^2}\frac{\partial z}{\partial x}\right) + \frac{\partial F}{\partial z}\left(\frac{\partial^2 z}{\partial x^2}\right)$$

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right)\frac{\partial z}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) =$$ enzovoort zie boven.

10-14 a) invullen $(1,3)$ in $(x,y)$ in vergelijkingen.

$1 + 9 - 2\ln uv + 2\ln 2 - 10 = 0 \Rightarrow uv = 2$

$1\ln u + 3\ln v - u^2 + 1 - 3\ln 2 = 0$ $\quad v = 2 \quad u = 1$

$$\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2\frac{1}{u} & \frac{2}{v} \\ \frac{x}{u} - 2u & \frac{y}{v} \end{vmatrix}_{(1,3,1,2)} = \begin{vmatrix} -2 & -1 \\ -1 & \frac{3}{2} \end{vmatrix} \neq 0$$

$f_1 f_2$ continue diff rond het gebied omgeving A.

IFS: In omgeving van $(x,y,u,v) = (1,3,1,2)$ is $u = u(x,y)$ $v = v(x,y)$

b) $\frac{\partial}{\partial x}(x^2 + y^2 - 2\ln uv + 2\ln 2 - 10) = 2x - 2\frac{1}{u}\cdot\frac{\partial u}{\partial x} - 2\frac{1}{v}\cdot\frac{\partial v}{\partial x} = 0$

$\frac{\partial}{\partial x}(f_2) = \ln u + \left(\frac{x}{u} - 2u\right)\frac{\partial u}{\partial x} + \frac{y}{v}\frac{\partial u}{\partial x} = 0$ Oplossen geeft uiteindelijk de gevraagde dingen.

c) $\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ krijg je door impliciet naar y te differentiëren

$\frac{\partial^2}{\partial x \partial y}$

$\frac{2}{u^2}\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial x} - \frac{2}{u}\frac{\partial u}{\partial x \partial y} + \frac{2}{u}\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} u_x u_y - \frac{2}{v^2}\frac{\partial^2 u}{\partial x \partial y} u_{xy} = 0$

$\frac{1}{u}u_y + \left(\frac{-x}{u^2}u_y - 2u_y\right)u_x + \left(\frac{x}{u} - 2u\right)u_{xy} + \left(\frac{v - y v_y}{v^2}\right)v_x + \frac{y}{v}v_{xy} = 0$

alleen $u_{xy}$ en $v_{xy}$ onbekend hoofddeterminant $\neq 0$

d $\tilde{f}(x,y) = f(x, y, u(x,y), v(x,y))$

taylor ontw. 1e orde $\tilde{f}(x,y) \approx \tilde{f}(a,b) + (x-a)\tilde{f}_x(a,b) + (y-b)\tilde{f}_y(a,b)$ $\quad \tilde{f}_x(x,y) = f_x + f_u u_x + f_v v_x$

$\tilde{f}_y(x,y) = f_y + f_u u_y + f_v v_y$

$f_x(x,y) = yuv + xyv\cdot u_x + xyu\cdot v_x = 6 + 6\cdot\frac{3}{7} + 3\cdot\frac{1}{2} = 12$

$f_y(x,y) = 2 + 6\cdot\frac{1}{7}(9 + \ln 2) + 3\cdot\frac{1}{2}(3 - \ln 2) = 20$

10-20 $f(p,V,T)=0 \quad V=V(p,T)$

[redacted]

$$\frac{\partial V(T,p)}{\partial T} = \frac{\partial V}{\partial T} + \frac{\partial V}{\partial p}\frac{\partial p}{\partial T}$$

a ~~$\frac{df}{dp} = \frac{\partial f}{\partial p} + \frac{\partial f}{\partial V}\cdot\frac{\partial V}{\partial p} + \frac{\partial f}{\partial T}\cdot\frac{\partial T}{\partial p}$~~

$$\frac{\partial T(p,V)}{\partial V} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial T} + \frac{\partial T}{\partial V}$$

toegepaste werkwijze
maak van $p = P(V,T)$ val in f inf.
$f(p(V,T),V,T)=0$ dit nu naar V en T differentieren-

$$\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial V} + \frac{\partial f}{\partial V} = 0 \qquad \frac{\partial p}{\partial V} = -\frac{\frac{\partial f}{\partial V}}{\frac{\partial f}{\partial p}}$$

$$\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial T} + \frac{\partial f}{\partial T} = 0 \qquad \frac{\partial p}{\partial T} = -\frac{\frac{\partial f}{\partial T}}{\frac{\partial f}{\partial p}}$$

Analoog $V = V(p,T)$ $f(p, V(p,T), T)$
en $T = T(p,V)$ $f(p, V, T(p,V))$
~~de eerste levert $\frac{\partial V}{\partial p}$ en $\frac{\partial V}{\partial T}$~~
~~de tweede levert~~

de eerste levert

$$\frac{\partial V}{\partial p} = -\frac{\frac{\partial f}{\partial p}}{\frac{\partial f}{\partial V}} \qquad \frac{\partial V}{\partial T} = -\frac{\frac{\partial f}{\partial T}}{\frac{\partial f}{\partial V}} \qquad \frac{\partial T}{\partial p} = -\frac{\frac{\partial f}{\partial p}}{\frac{\partial f}{\partial T}} \qquad \frac{\partial T}{\partial V} = -\frac{\frac{\partial f}{\partial V}}{\frac{\partial f}{\partial T}}$$

$$\frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial p} = -\frac{\frac{\partial f}{\partial V}}{\frac{\partial f}{\partial p}}\cdot -\frac{\frac{\partial f}{\partial T}}{\frac{\partial f}{\partial V}}\cdot -\frac{\frac{\partial f}{\partial p}}{\frac{\partial f}{\partial T}} = -1$$

b1 f moet een maal kontinue differentieerbaar.
$\frac{\partial f}{\partial V}, \frac{\partial f}{\partial T}, \frac{\partial f}{\partial p}$ moeten ongelijk zijn aan nul.
betrekking $f(p,V,T)=0$ moet gelden in een omgeving van een zekerpunt (voorw voor IFS)

b2 $pV - kT = 0 \Rightarrow$ alles klopt.

11-1 a f uniform continue op verz D
Als $\forall \varepsilon>0 \ \exists \delta>0 \ \forall (x,y)\in D \ \|x-y\|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

$f(x_1,x_2) = 2x_1$
$|f(x)-f(y)| = |2x_1 - 2y_1| = 2|x_1-y_1| = 2\sqrt{(x_1-y_1)^2} \le 2\sqrt{(x_1-y_1)^2+(x_2-y_2)^2} < 2\cdot\delta = \varepsilon$
voor gegeven $\varepsilon$ kies nu $\delta = \frac{\varepsilon}{2}$
Dus voor gegeven $\varepsilon>0$ impliceert $d(x,y)<\delta = \frac{\varepsilon}{2}$ dat $|f(x)-f(y)|<\varepsilon$

UNIFORM CONTINUE $\Rightarrow$ CONTINUE
omgekeerde geldt met een zin op kompacten (gesloten, begrensd) verz.
kompact domein f continue $\Rightarrow$ f uniform kontinue.

b $g(x_1,x_2) = x_1 x_2$ $G = \{(x_1,x_2) : |x_1|<2, |x_2|<3\}$
we merken op dat g ook continue is op $\bar{G} = \{(x_1,x_2) : |x_1|\le 2, |x_2|\le 3\}$
g uniform continue op $\bar{G}$ dus ook op G zelf.
$|f(x)-f(y)| = |x_1 x_2 - y_1 y_2|$ ~~$\sqrt{(x_1x_2 - y_1y_2)^2}$~~ $\le \sqrt{x_1^2x_2^2 - 2x_1y_1x_2y_2 + y_1^2y_2^2}$

[redacted]
$\sqrt{x_1^2 - 2x_1y_1 + y_1^2 + x_2^2 + y_2^2}$

$|(x_1 - y_1 + y_1)x_2 - y_1 y_2| = |(x_1-y_1)x_2 + y_1x_2 - y_1y_2)| =$

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$$|(x_1-y_1)x_2+y_2(x_2-y_1)| \le (\Delta \text{ong}) \le |(x_1-y_1)x_2| + |y_2(x_2-y_2)| = |x_1-y_1||x_2| +$$
$$|y_2||x_2-y_2| \le 3|x_1-y_1| + 2|x_2-y_2| \le 5\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$

kies nu $\delta = \frac{\varepsilon}{5}$ dan $d(x,y)<\delta \Rightarrow |g(x)-g(y)|<\varepsilon$

11-20 $\quad I(x) = \int_{x^2}^{e^{2x}} xy\, dy$

$$\frac{dI(x)}{dx} = \int_{x^2}^{e^{2x}} y\,dy + x\cdot e^{2x}\cdot 2e^{2x} - x^3\cdot 2x = \int_{x^2}^{e^{2x}} y\,dy + 2x\cdot e^{4x} - 2x^4$$

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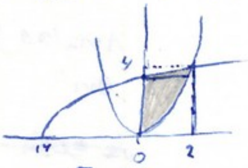
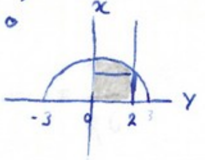
11-5a $\quad \int_0^2 \left\{ \int_{x^2}^{\sqrt{x+14}} f(x,y)\,dy \right\} dx$



$\sqrt{14}$

$$= \int_0^4 \left\{ \int_{\max\{y^2-14,0\}}^{\sqrt{y}} f(x,y)\,dx \right\} dy = \int_0^{\sqrt{14}} \left\{ \int_0^{\sqrt{y}} f(x,y)\,dx \right\} dy + \int_{\sqrt{14}}^{4} \left\{ \int_{y^2-14}^{\sqrt{y}} f(x,y)\,dx \right\} dy$$

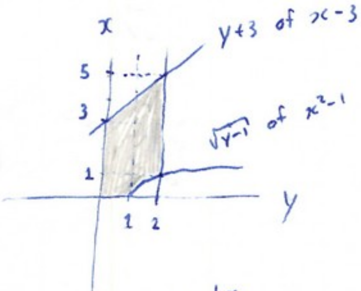
11-5b $\quad \int_0^2 \left\{ \int_0^{\sqrt{9-y^2}} f(x,y)\,dx \right\} dy$



$\sqrt{5}$

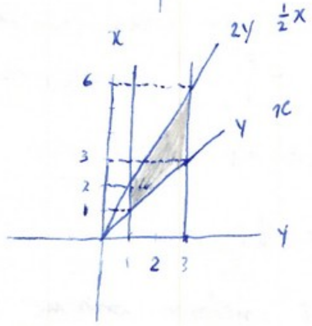
$$= \int_0^{\sqrt{5}} \left\{ \int_0^2 f(x,y)\,dy \right\} dx + \int_{\sqrt{5}}^{3} \left\{ \int_0^{\sqrt{9-x^2}} f(x,y)\,dy \right\} dx$$

11-5c $\quad \int_0^1 \left\{ \int_0^{y+3} f(x,y)\,dx \right\} dy + \int_1^2 \left\{ \int_{\sqrt{y-1}}^{y+3} f(x,y)\,dx \right\} dy =$



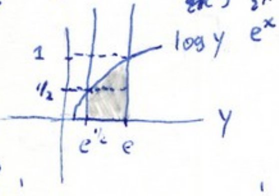
$$\int_0^1 \left\{ \int_0^{x^2-1} f(x,y)\,dy \right\} dx + \int_1^3 \left\{ \int_0^2 f(x,y)\,dy \right\} dx + \int_3^5 \left\{ \int_{x-3}^{2} f(x,y)\,dy \right\} dx$$

11-5d $\quad \int_1^3 \left\{ \int_y^{2y} f(x,y)\,dx \right\} dy =$



$$\int_1^2 \left\{ \int_1^x f(x,y)\,dy \right\} dx + \int_2^3 \left\{ \int_{\frac{1}{2}x}^{x} f(x,y)\,dy \right\} dx + \int_3^6 \int_{\frac{1}{2}x}^{3} f(x,y)\,dy \Big\} dx$$

11-5g $\quad \int_{\sqrt{e}}^{e} \left\{ \int_0^{\log y} f(x,y)\,dx \right\} dy$



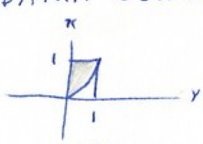
$$= \int_0^{1/2} \left\{ \int_{e^{1/2}}^{e} f(x,y)\,dy \right\} dx + \int_{1/2}^{1} \left\{ \int_{e^x}^{e} f(x,y)\,dy \right\} dx$$

11-6d $\quad \int_{1/2}^1 \left\{ \int_0^1 \frac{x}{\sqrt{x^2+y}}\,dx \right\} dy = \int_{1/2}^1 \left[\sqrt{x^2+y}\right]_0^1 dy = \int_{1/2}^1 \sqrt{1+y} - \sqrt{y}\,dy = \frac{2}{3}\left[(1+y)^{3/2} - (y)^{3/2}\right]_{1/2}^1 = \frac{2}{3}\left(2^{3/2} - 1^{3/2} - \tfrac{3}{2}^{3/2} + \tfrac{1}{2}^{3/2}\right)$

11-6e $\quad \int_2^3 \left\{ \int_0^1 (3x-y)^{10}\,dx \right\} dy = \int_2^3 \frac{1}{33}\left[(3x-y)^{11}\right]_0^1 dy = \frac{1}{33}\int_2^3 (1-y)^{11} - (y)^{11}\,dy = -\frac{1}{33}\cdot\frac{1}{12}\left[(1-y)^{12} - (y)^{12}\right]_2^3 =$

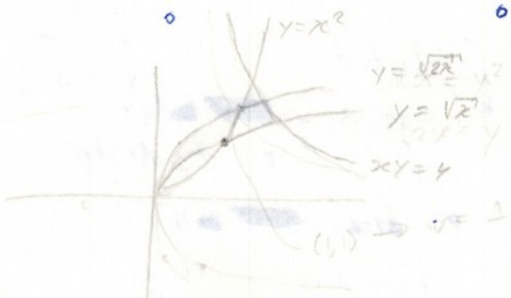
$$-\frac{1}{33\cdot 12}\left((-2)^{12} - (3)^{12} - (-1)^{12} + (2)^{12}\right)$$

11-8 $\int_0^1\left\{\int_{y^2}^1 y\frac{\sin x}{x}\,dx\right\}dy$



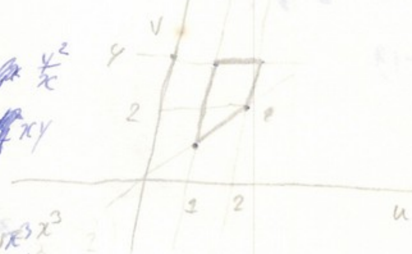
$= \int_0^1\left\{\int_0^{\sqrt{x}} y\frac{\sin x}{x}\,dy\right\}dx = \frac{1}{2}\int_0^1\left[y^2\cdot\frac{\sin x}{x}\right]_0^{\sqrt{x}}dx = \frac{1}{2}\int_0^1 \sin x\,dx = \frac{1}{2}\left[-\cos x\right]_0^1 = -\frac{1}{2}(\cos 1 + 1)$

11-14




$u = \frac{y^2}{x}$

$v = xy$



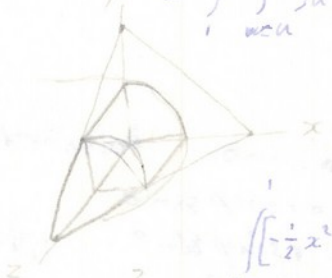

$y = x^2$, $u = xy = x^3$, $u = \frac{x^2}{x}$, $u = \sqrt{x}\,x^2 y^2 = \sqrt{x^3}\,x^3$, $u = v$

$\iint_B xy\,dx\,dy = \qquad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} \qquad \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} -y^2/x^2 & 2y/x \\ y & x\end{vmatrix} = \left|\frac{-3y^2}{x}\right| = |-3u| = 3u$

$\iint_B xy\,dxy = \iint_B v\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv = \iint_B \frac{v}{3u}\,du\,dv$ of:

$= \int_1^2\int_{v=u}^4 \frac{v}{3u}\,dv\,du = \int_1^2\left[\frac{1}{6}\frac{v^2}{u}\right]_u^4 du = \int_1^2\left(\frac{8}{3u} - \frac{1}{6}u\right)du = \left[\frac{8}{3}\ln u - \frac{1}{12}u^2\right]_1^2 = \frac{8}{3}\ln 2 - \frac{1}{4}$

11-11

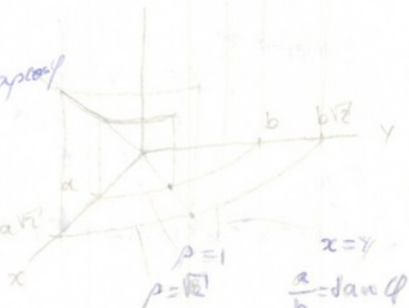



$\int_0^1\int_0^{1-y^2}\int_0^{2-x-y} y\,dz\,dx\,dy = \int_0^1\int_0^{1-y^2}(2-x-y)\cdot y\,dx\,dy =$

$\int_0^1\left[-\frac{1}{2}x^2 + xy(2-y)\right]_0^{1-y^2}dy = \int_0^1 -\frac{1}{2}(1-y^2) + (1-y^2)y(2-y) = \frac{17}{60}$

11-12

$\xi = 3a\rho\cos\varphi$




$x = a\rho\cos\varphi$

$y = b\rho\sin\varphi$

$z = \xi$

$\begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\xi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\xi}\end{vmatrix} = \begin{vmatrix} a\cos\varphi & -a\rho\sin\varphi & 0 \\ b\sin\varphi & b\rho\cos\varphi & 0 \\ 0 & 0 & 1\end{vmatrix} = ab\rho$

$\rho = 1$ $\quad x = y$

$\rho = \sqrt{2}$ $\quad \frac{a}{b} = \tan\varphi$

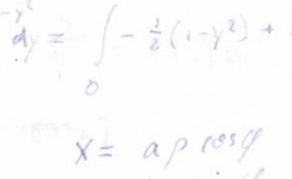
$\varphi = \arctan\frac{a}{b}$

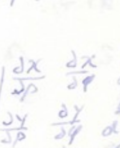
$\int_{\varphi=0}^{\arctan a/b}\int_{\rho=1}^{\sqrt{2}}\int_0^{3a\rho\cos\varphi}\left|\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\xi)}\right|d\xi\,d\rho\,d\varphi =$

$\int_{\varphi=0}^{\arctan a/b}\int_1^{\sqrt{2}}\int_0^{3a\rho\cos\varphi} ab\rho\,d\xi\,d\rho\,d\varphi = \int_0^{\arctan a/b}\int_1^{\sqrt{2}} 3a^2b\rho^2\cos\varphi\,d\rho\,d\varphi = \int_0^{\arctan a/b}\left[a^2b\rho^3\cos\varphi\right]_1^{\sqrt{2}}d\varphi =$

$\int_{\varphi=0}^{\arctan a/b} a^2b(2\sqrt{2}-1)\cos\varphi\,d\varphi = a^2b(2\sqrt{2}-1)\sin\varphi\Big|_0^{\arctan a/b} = \frac{a^3b\,(2\sqrt{2}-1)}{\sqrt{a^2+b^2}}$

DATUM 19 oktober 1982.




$\iiint yz\,dx\,dy\,dz$

transformatie polaire coördinaten.

$x = r\cos\varphi\sin\theta$

$y = r\sin\varphi\sin\theta \qquad |J| = r^2\sin\theta$

$z = r\cos\theta$

$\int_{\varphi=0}^{\frac{\pi}{2}}\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=0}^{1} r^4\sin\varphi\sin\theta^2\cos\theta\,dr\,d\theta\,d\varphi$

$= \frac{1}{5}\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\varphi\cos\theta\sin\theta\,d\theta\,d\varphi = \frac{1}{5}\cdot\frac{1}{3}\int_{\varphi=0}^{\frac{\pi}{2}}\sin\varphi\left[\sin^3\theta\right]_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}d\varphi$

$\frac{1}{15}\left(1-\frac{1}{\sqrt{8}}\right)\left[-\cos\varphi\right]_0^{\frac{\pi}{2}} = \frac{1}{15}\left(1-\frac{1}{\sqrt{8}}\right)$ alternatief z.o.z.

Cylinder coördinaten

$x = r\cos\varphi$

$y = r\cos\varphi$

$z = \xi$

$$\frac{1}{2}\sqrt{2}\ \sqrt{1-\xi^2}$$

$$\int_{\varphi=0}^{\frac{\pi}{2}}\int_{\xi=0}\int_{r=\xi}\xi\, r\cos\varphi$$

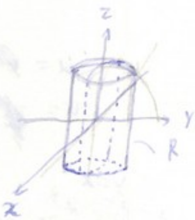
$x = r\cos\varphi\sin\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\theta$

11-19





$$\iiint_R dx\,dy\,dz$$

$z = z$
$x = r\cos\varphi$
$y = r\sin\varphi$

$$\int_{\varphi=0}^{2\pi}\int_{r=a}^{a}\int_{z=[redacted]\sqrt{4a^2-r^2}}^{z=[redacted]\sqrt{4a^2-r^2}} r\ dz\,dr\,d\varphi$$

$x^2+y^2+z^2 = 4a^2$

$r^2 + z^2 = 4a^2$

$z = \sqrt{4a^2-r^2}$

[redacted]

$$\int_0^{2\pi}\int_0^{a} 2r\sqrt{4a^2-r^2}\,dr\,d\varphi = \int_0^{2\pi}\left[\frac{2}{3}(4a^2-r^2)^{3/2}\right]_0^a d\varphi =$$

$r^2 = 4\sin t$

$$-\frac{2}{3}\int_0^{2\pi}\left((3a^2)^{3/2}-(4a^2)^{3/2}\right)d\varphi = -\frac{4}{3}\pi\,(3^{3/2}-4^{3/2})a^3 = \frac{4}{3}\pi\,(8-3^{3/2})a^3$$

11-21

tot. massa $= \iiint_{bol} f(x,y,z)\,dx\,dy\,dz$

Afstand tot de oorsprong r (van de (x, y, z))

dan $\rho(r) = \alpha r + \beta$

$\rho(0) = 2 \qquad \beta = 2$

$\rho(2a) = 1 \qquad \alpha = \frac{1}{2a}$

$\rho(r) = -\frac{r}{2a} + 2$

transf. mbv bolkoördinaten

$x = r\cos\varphi\sin\theta$

$y = r\sin\varphi\sin\theta \qquad |J| = r^2\sin\theta$

$z = r\cos\theta$

integraal:

kleine bol $x^2+y^2+(z-a)^2 = a^2$

$r^2\sin^2\theta + r^2\cos^2\theta - 2ar\cos\theta = 0$

$r^2 = 2ar\cos\theta$

$r = 2a\cos\theta$

$$\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{2a\cos\theta}\left(-\frac{r}{2a}+2\right) r^2\sin\theta\ dr\,d\theta\,d\varphi =$$

$$\int_0^{2\pi}\int_0^{\pi/2}\left[-\frac{1}{2a}r^4 + \frac{2}{3}r^3\right]^{2a\cos\theta}\sin\theta\,d\theta\,d\varphi = \int_0^{2\pi}\int_0^{\pi/2}\left(-2a^3\cos^4\theta\sin\theta + \frac{16}{3}a^3\cos^3\theta\sin\theta\right)d\theta\,d\varphi$$

$$\int_{\varphi=0}^{2\pi}\left[\frac{2a^3}{5}\cos^5\theta - \frac{4}{3}a^3\cos^4\theta\right]_0^{\pi/2} d\varphi = \int_0^{2\pi}\left(-\frac{2}{5}a^3 + \frac{4}{3}a^3\right)d\varphi = 2\pi a^3\left(-\frac{2}{5}+\frac{4}{3}\right) = \frac{28}{15}\pi a^3$$

alternatieven: 1) volgorde $\theta, r$ verwisselen 2) cylindercoördinaten (nog eenvoudig)
3) met bolcoördinaten.

22 c) $\iint \frac{1}{(x^2+y^2)^{3/2}}\,dx\,y$

bestaat niet.



$x = r\cos\varphi$
$y = r\sin\varphi$
$0 \le \varphi \le \pi/2$
$\varepsilon \le r \le 1$

$(\underline{a}+\underline{b}, (\underline{a}\times\underline{c})\times(\underline{a}+\underline{b})) = (\underline{a}+\underline{b}, ((\underline{a}\times\underline{c})\times\underline{a}) + ((\underline{a}\times\underline{c})\times\underline{b})) =$

$(\underline{a}, ((\underline{a}\times\underline{c})\times\underline{a})) + (\underline{a}, ((\underline{a}\times\underline{c})\times\underline{b})) + (\underline{b}, ((\underline{a}\times\underline{c})\times\underline{a})) + (\underline{b}, ((\underline{a}\times\underline{c})\times\underline{b}))$

12.3a $(\underline{a}+\underline{b}, (\underline{a}\times\underline{c})\times(\underline{a}+\underline{b})) = ((\underline{a}\times\underline{c}), (\underline{a}+\underline{b})\times(\underline{a}+\underline{b})) = (\underline{a}\times\underline{c}, \underline{o}) = 0$

12.3b $\underline{a}\times(\underline{a}\times(\underline{a}\times\underline{b})) = (\underline{a}, (\underline{a}+\underline{b}))\underline{a} - (\underline{a},\underline{a})(\underline{a}\times\underline{b}) =$

$(\underline{a},\underline{a})\underline{a} + (\underline{a},\underline{b})\underline{a} - (\underline{a},\underline{a})(\underline{a}\times\underline{b}) = (\underline{a},\underline{a})(\underline{a} - (\underline{a}\times\underline{b})) + (\underline{a},\underline{b})\underline{a}$

$= \underline{a}\times\left((\underline{a},\underline{b})\underline{a} - (\underline{a},\underline{a})\underline{b}\right) = \underline{a}\times((\underline{a},\underline{b})\underline{a}) - \underline{a}\times(\underline{a},\underline{a})\underline{b} =$

$\underline{o} - \underline{a}\times(\underline{a},\underline{a})\underline{b} = (\underline{a},\underline{a})\underline{b}\times\underline{a}$

12.3c $(\underline{a}\times\underline{b}, \underline{c}\times\underline{d}) =$

$(\underline{c}, \underline{d}\times(\underline{a}\times\underline{b})) = (\underline{c}, (\underline{d},\underline{b})\underline{a} - (\underline{d},\underline{a})\underline{b}) = (\underline{c},\underline{a})(\underline{d},\underline{b}) -$
$(\underline{c},\underline{b})(\underline{d},\underline{a})$

12.3d $\underline{a}\times(\underline{b}\times\underline{c}) + \underline{b}\times(\underline{c}\times\underline{a}) + \underline{c}\times(\underline{a}\times\underline{b}) = 0$

$(\underline{a},\underline{c})\underline{b} - (\underline{a},\underline{b})\underline{c} + (\underline{b},\underline{a})\underline{c} - (\underline{b},\underline{c})\underline{a} + (\underline{c},\underline{b})\underline{a} - (\underline{c},\underline{a})\underline{b} = \underline{o}$

12.5a $(\underline{a},\underline{b}) \neq 0$

- $\underline{a}\times\underline{b}$ staat loodrecht op het vlak voort gebracht door

$\underline{x} = \tau\underline{a} + \nu\underline{b} + \mu(\underline{a}\times\underline{b})$ $\quad \mu = 0$

$(\underline{a}\times\underline{b}, \underline{x}) = 0 \Rightarrow$ $\underline{x}$ staat loodrecht op $\underline{a}\times\underline{b}$

$(\underline{a}\times\underline{b}, \tau\underline{a} + \nu\underline{b}) = (\underline{a}\times\underline{b}, \tau\underline{a}) + (\underline{a}\times\underline{b}, \nu\underline{b}) + (\underline{a}\times\underline{b}, \mu(\underline{a}\times\underline{b})) = 0$

$= \tau(\underline{a}, \underline{a}\times\underline{b}) + \nu(\underline{b}, \underline{a}\times\underline{b}) = \tau\cdot 0 + \nu\cdot 0 \quad (\underline{a}\times\underline{b}, \underline{a}\times\underline{b})\cdot\mu = 0$

dan moet $\mu = 0$ en $\tau$ en $\nu$ mogen willekeurige waarde hebben.

12.8a $r = \|\underline{r}\| = \sqrt{r_1^2 + r_2^2 + r_3^2}$ als $\underline{r} = (r_1, r_2, r_3)$

$$\frac{dr}{du} = \frac{2r_1\frac{dr_1}{du} + 2r_2\frac{dr_2}{du} + 2r_3\frac{dr_3}{du}}{2\sqrt{r_1^2+r_2^2+r_3^2}} = \frac{(\underline{r}, \frac{d\underline{r}}{du})}{r} = \frac{(\underline{r}, \frac{d\underline{r}}{du})}{\|\underline{r}\|}$$

alternatief $r^2 = \langle \underline{r}, \underline{r}\rangle$ diff $2r\frac{dr}{du} = 2\langle \underline{r}, \frac{d\underline{r}}{du}\rangle$

$\frac{dr}{du} = \frac{\langle \underline{r}, \frac{d\underline{r}}{du}\rangle}{r}$ ← is bijzondere geval van de produktregel voor het diff. van inprodukten

$$\frac{d}{du}(\underline{a},\underline{b}) = \left(\frac{d\underline{a}}{du}, \underline{b}\right) + \left(\underline{a}, \frac{d\underline{b}}{du}\right)$$

b $\frac{d}{du}(r^3\underline{r} + \underline{p}\times\underline{r}) = 3r^2\frac{dr}{du}\cdot\underline{r} + r^3\frac{d\underline{r}}{du} + \underline{o} + \underline{p}\times\frac{d\underline{r}}{du}$

$\frac{d}{du}\|\underline{r} + r\underline{q}\|^2 = \frac{d}{du}(\underline{r} + r\underline{q}, \underline{r} + r\underline{q}) = \frac{d}{du}(\underline{r},\underline{r}) + 2\frac{d}{du}(\underline{r}, r\underline{q}) + \frac{d}{du}(r\underline{q}, r\underline{q})$

$= 2(\underline{r}, \frac{d}{du}\underline{r}) + 2(\frac{d}{du}\underline{r}, r\underline{q}) + 2(\underline{r}, \frac{dr}{du}\underline{q}) + 2(\frac{dr}{du}\underline{q}, r\underline{q})$

12.9

$$\frac{d}{dt}\left(\underline{r}, \frac{d\underline{r}}{dt} \times \frac{d^2\underline{r}}{dt^2}\right) = \left(\frac{d\underline{r}}{dt}, \frac{d\underline{r}}{dt} \times \frac{d^2\underline{r}}{dt^2}\right) + \left(\underline{r}, \frac{d^2\underline{r}}{dt^2} \times \frac{d^2\underline{r}}{dt^2}\right) + \left(\underline{r}, \frac{d\underline{r}}{dt} \times \left(\frac{d^3\underline{r}}{dt^3} = 4\underline{r}\right)\right) =$$

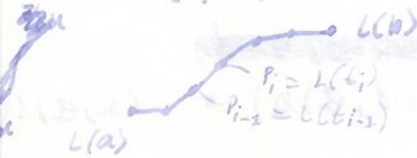
$$\frac{d}{dt} f(t) = 0 \qquad f(0) = \left(\underline{0}, \frac{d\underline{r}}{dt} \times \frac{d^2\underline{r}}{dt^2}\right) = 0$$

uit $f(0)=0$ en $\frac{d}{dt} f(t) = 0$ volgt $f(t) \equiv 0$

12-10 $x = a(\cos u + \log(\tan \frac{u}{2}))$

$y = a(1 - \sin u)$

met $u \in [0, \pi]$ en $a > 0$, a konstant.




$L[a,b] \to \mathbb{R}^2$

$$\sum_i \| L(t_i) - L(t_{i-1}) \| = \sum_i \left\| \frac{L(t_i) - L(t_{i-1})}{t_i - t_{i-1}} \right\| (t_i - t_{i-1})$$

$$\approx \int_a^b \left\| \frac{dL}{dt} \right\| dt$$

$$\underline{L}(u) = a \begin{bmatrix} \cos u + \log(\tan \frac{u}{2}) \\ 1 - \sin u \end{bmatrix}, a > 0$$, eerst tan $\frac{u}{2}$ herschrijven dan dubbele hoek

$$\frac{d\underline{L}(u)}{du} = a \begin{bmatrix} -\sin u + \frac{1}{\tan u/2} \cdot \frac{1}{\cos^2 u/2} \cdot \frac{1}{2} \\ \cos u \end{bmatrix} = \begin{bmatrix} -\sin u + \frac{1}{\sin u} \\ -\cos u \end{bmatrix}$$

$$\left\| \frac{d}{du} \underline{L}(u) \right\| = \sqrt{\left(-\sin u + \frac{1}{\sin u}\right)^2 + \cos^2 u} = \sqrt{\sin^2 u + \frac{1}{\sin^2 u} + \cos^2 u} = \sqrt{\frac{1}{\sin^2 u} - 1} =$$

$$\sqrt{\frac{1 - \sin^2 u}{\sin^2 u}} = \sqrt{\frac{\cos^2 u}{\sin^2 u}}$$ booglengte:

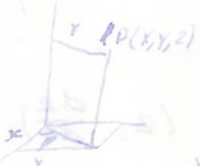
$$\int_{1/8\pi}^{7/8\pi} \left| \frac{\cos u}{\sin u} \right| du = \int_{\pi/8}^{\pi/2} \frac{\cos u}{\sin u} du - \int_{\pi/2}^{7/8\pi} \frac{\cos u}{\sin u} du = \left[\log(\sin u)\right]_{\pi/8}^{\pi/2} - \left[\log(\sin u)\right]_{\pi/2}^{7/8\pi} =$$

$$-\log \sin \frac{\pi}{8} - \log \sin \tfrac{7}{8}\pi = -2 \log \sin \frac{\pi}{8} = -\log \sin^2 \frac{\pi}{8} = -\log\left(\tfrac{1}{2} - \tfrac{1}{2}\cos \tfrac{\pi}{4}\right)$$

$$= \log\left(\tfrac{1}{2} - \tfrac{1}{4}\sqrt{2}\right)$$

DATUM: 021182

12-12




$r = \varphi$ $\qquad r^2 = x^2 + x^2 \tan^2\varphi = x(1 + \tan^2\varphi) = \frac{x^2}{\cos\varphi}$

$\tan\varphi = \frac{y}{x}$ $\qquad x^2 = r^2 \cos^2\varphi$

$r^2 = x^2 + y^2$ $\qquad x = r\cos\varphi$ $\qquad x = \varphi\cos\varphi$

$y = x\tan\varphi$ $\qquad y = r\sin\varphi$ $\qquad y = \varphi\sin\varphi$

$t = r^2$ $\qquad z = \varphi^2$

$\rho(\varphi) = \varphi$,

$\int_K \rho\, ds = \int_0 \varphi \cdot \| L'(\varphi) \| \, d(\varphi)$, waarbij L de parametrisering van K is

$$L'(\varphi) = \begin{bmatrix} \cos\varphi - \varphi\sin\varphi \\ \sin\varphi + \varphi\cos\varphi \\ 2\varphi \end{bmatrix} \Rightarrow \| L'(\varphi) \| = \sqrt{\cos^2\varphi - 2\varphi\cos\varphi\sin\varphi + \varphi^2\sin^2\varphi + \sin^2\varphi + 2\varphi\sin\varphi\cos\varphi + \varphi^2\cos^2\varphi + 4\varphi^2} =$$

$$\sqrt{1 + 5\varphi^2} = \text{etc}$$

werk college: voortgezette WISKUNDE I        DATUM 02-11-82  BLAD 5

13-1 a  $\text{grad } f = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix}$   $\text{grad } f(1,1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

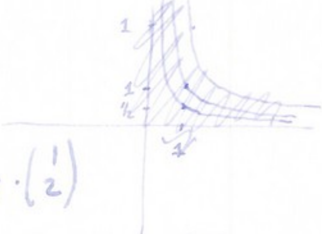
b  [illegible] $(\text{grad } f, \begin{bmatrix} 1 \\ 2 \end{bmatrix}) = 4$   $\dfrac{(\text{grad } f, \begin{bmatrix} 1 \\ 2 \end{bmatrix})}{\left\|\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\|} = \dfrac{4}{\sqrt{5}} = \frac{4}{5}\sqrt{5}$

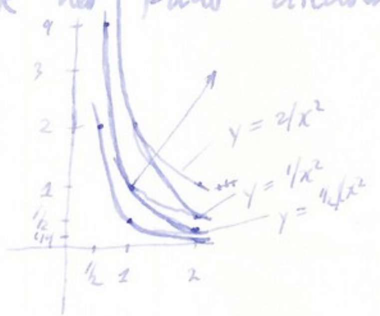
c  $f(1,1) = 1$   $f(2,3) \approx f(1,1) +$ [illegible] $4 = 5$  } verschil 7.
$f(2,3) = 12$

$f(\frac{3}{2},2) \approx f(1,1) + \frac{1}{2} \cdot 4 = 3$  } $3/2$
$f(\frac{3}{2},2) = \frac{9}{4} \cdot 2 = \frac{9}{2} = 4\frac{1}{2}$
$f(\frac{5}{4},\frac{3}{2}) \approx f(1,1) + \frac{1}{4} \cdot 4 = 2$   $f(\frac{5}{4},\frac{3}{2}) = \frac{25 \cdot 9}{16 \cdot 2} = \frac{225}{32}$ } verschil $\frac{11}{32}$
verschil wordt kleiner naarmate het punt dichter bij (1,1) komt.

d  $x^2 y = c \Rightarrow y = \frac{c}{x^2}$   $y = \frac{1/2}{x^2}$

$y$  1  $1/4$






e  $\underline{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{a}{\sqrt{5}} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$f(\underline{A}) = f(1 + \frac{a}{\sqrt{5}}, 1 + \frac{2a}{\sqrt{5}}) = (1 + \frac{a}{\sqrt{5}})^2 (1 + \frac{2a}{\sqrt{5}}) = (1 + \frac{2a}{\sqrt{5}} + \frac{a^2}{5})(1 + \frac{2a}{\sqrt{5}}) =$
$(1 + \frac{2a}{\sqrt{5}} + \frac{a^2}{5}) + (\frac{2a}{\sqrt{5}} + \frac{4a^2}{5} + \frac{2a^3}{5\sqrt{5}}) = 1 + \frac{4a}{\sqrt{5}} + a^2 + \frac{2a^3}{5\sqrt{5}}$
$f(\underline{A}) = f(1,1) + \frac{a}{\sqrt{5}} \cdot 4 = 1 + \frac{4a}{\sqrt{5}}$

raakvlak $3x + 2y + 6z = -1$
$(x_1, \text{grad } f(x_0)) = (x_0, \text{grad } f(x_0))$
raakvlak door $x_0$

13-2 a  def $f(x,y,z) = x^2 + 2y^2 - 3z^2$   $\text{grad } f = \begin{bmatrix} 2x \\ 4y \\ 6z \end{bmatrix}$
$\text{grad } f(3,1,-2) = \begin{bmatrix} 6 \\ 4 \\ +12 \end{bmatrix} =$ [illegible]  genormeerd  $\begin{bmatrix} 3 \\ 2 \\ +6 \end{bmatrix} \dfrac{1}{\sqrt{9+4+36}} = \frac{1}{7}\begin{bmatrix} 3 \\ 2 \\ +6 \end{bmatrix}$

13-3 c  ~~theoretisch gezien is grad f = $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ maar dit is natuurlijk onzin daar $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ geen lengte heeft. Dus hij bestaat niet.~~  $\text{grad } f = (\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}})$
voor (0,0,0) bestaat deze niet

13-5  $f(x,y,z) = 2x^2 - 2xy + 2y^2 - 2zy + 2z^2 - 2zx = 2(x^2 + y^2 + z^2 - xy - zy - zx)$

$\text{grad } f = 2\begin{bmatrix} 2x - y - z \\ 2y - x - z \\ 2z - x - y \end{bmatrix}$   $\text{grad } f(2,1,3) = 2\begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$ [illegible]

$f'(f; \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}) = 2\begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 2(-9+6) = -6$

b  in richting $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

vrijdag 19 nov 8:30
Hier

(wij missen de heer Bragert)

## koordinatie transformaties.

Kettingregel. bij dif. van functies. functies zijn gecontinue diff. baar.

$f(x)$ op $(a,b)$ $g(t)$ op $(\alpha,\beta)$ stel $f(x) \in (\alpha,\beta)$

$$\underbrace{x \in (a,b) \xrightarrow{f} f(x) = t \in (\alpha,\beta) \xrightarrow{g} g(t)}_{g^* = g \circ f} = g(f(x))$$

$$\frac{dg^*}{dx} = \frac{dg}{dt} \cdot \frac{df}{dx} \qquad t = f(x)$$

$\underbrace{f_1(x), f_2(x)}_{f(x)}$ op $(a,b)$ $g(\underbrace{t_1, t_2}_{t})$ op $V \subset \mathbb{R}^2$ stel $f(x) \in V$

$$\frac{dg^*}{dx} = \frac{\partial g}{\partial t_1}\frac{df_1}{dx} + \frac{\partial g}{\partial t_2}\frac{df_2}{dx} \quad \text{met } t = f(x)$$

$$\underbrace{x \in (a,b) \xrightarrow{f} f(x) = t \in (\alpha,\beta) \xrightarrow{g} g(t)}_{g^* = g \circ f} = g(f(x))$$

$\underbrace{f_1, f_2, \cdots f_m}_{f}$ van $\underbrace{x_1, x_2 \cdots x_n}_{x}$ $\mathbb{R}^n \to \mathbb{R}^m$ $\underbrace{g_1, g_2 \cdots g_p}_{g}$ van $\underbrace{t_1, t_2 \cdots t_m}_{t}$

stel $f(u) \in V$

$$\underbrace{\underbrace{x \in U \xrightarrow{f} f(x) = t \in V \xrightarrow{g} g(t)}_{g^* = g \circ f}}_{g_1^*, g_2^* \cdots g_p^*}$$

$$\frac{dg_i^*}{dx_j} = \frac{\partial g_i}{\partial t_1}\frac{df_1}{dx_j} + \frac{\partial g_i}{\partial t_2}\frac{df_2}{dx_j} + \cdots + \frac{\partial g_i}{\partial t_m}\frac{df_m}{dx_j} = \sum_{k=1}^{m} \frac{\partial g_i}{\partial t_k}\frac{df_k}{dx_j}$$

$$\begin{bmatrix} \frac{\partial g_1^*}{\partial x_1} & \cdots & \frac{\partial g_1^*}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_p^*}{\partial x_1} & \cdots & \frac{\partial g_p^*}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial t_1} & \cdots & \frac{\partial g_1}{\partial t_m} \\ \vdots & & \vdots \\ \frac{\partial g_p}{\partial t_1} & \cdots & \frac{\partial g_p}{\partial t_m} \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$$\frac{\partial^2 g_i^*}{\partial x_k \partial x_j} = \text{zie boek.} = \frac{\partial}{\partial x_k}\frac{dg_i^*}{dx_j} = \frac{\partial}{\partial x_k}\left(\sum \frac{\partial g_i}{\partial t_\mu}\frac{\partial f_\mu}{\partial x_j}\right) = \sum \frac{\partial^2 g_i}{\partial t_\mu \partial t_l}\frac{\partial f_l}{\partial x_k}\frac{\partial f_\mu}{\partial x_j} + \sum \frac{\partial f_i}{\partial t_\mu}\frac{\partial^2 f_\mu}{\partial j\, \partial x_k}$$

### Inverse functie stelling

$y = f(x)$ $f'(x) \neq 0$ in $x = a$

dan bestaat er een omgeving rond a waar f inverteerbaar is. $(\alpha,\beta)$

$f((\alpha,\beta)) \to W$ $f^{-1}$ $y \in W \xrightarrow{f^{-1}} x \in (\alpha,\beta)$, zodat $f(x) = y$.

$$\frac{df}{dx} \cdot \frac{df^{-1}}{dy} = 1$$

bij $f'(x) = 0$ kan je er niets over zeggen.

$\underbrace{f_1, f_2 \cdots f_n}_{f}$ van $\underbrace{x_1, x_2 \cdots x_n}_{x}$

$$D_x(f) = \frac{\partial(f_1, f_2, \cdots f_n)}{\partial(x_1, x_2 \cdots x_n)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad \det \neq 0 \text{ in } x = a$$

er is omg. V van a waarop f inverteerbaar is.

$Y \in W = f(v)$ $f^{-1}: y \to x$ zodat $y = f(x)$

$$D_x(f)\, D_y(f^{-1}) = \frac{\partial(f_1 \cdots f_n)}{\partial(x_1 \cdots x_n)} \frac{\partial(f_1^{-1} \cdots f_n^{-1})}{\partial(y_1 \cdots y_n)} = \frac{\partial(y_1, \cdots y_n)}{\partial(x_1 \cdots x_n)} \frac{\partial(x_1 \cdots x_n)}{\partial(y_1 \cdots y_n)} = E$$

vraagstuk 2

$y_1 = f_1(x) = x_1^2 + x_1x_2$

$y_2 = f_2(x) = x_1x_2 + x_2^2$ } zeer domme functies.

$$D_x y = \begin{bmatrix} 2x_1+x_2 & x_1 \\ x_2 & x_1+2x_2 \end{bmatrix} = 2x_1^2+5x_1x_2+2x_2^2-x_1x_2 = 2(x_1+x_2)^2 \neq 0$$

voor $x_1+x_2 \neq 0$

$$\begin{bmatrix} 2x_1+x_2 & x_1 \\ x_2 & x_1+2x_2 \end{bmatrix}\begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(2x_1+x_2)\frac{\partial x_1}{\partial y_1} + x_1\frac{\partial x_2}{\partial y_1} = 1$
$x_2\frac{\partial x_1}{\partial y_1} + (x_1+2x_2)\frac{\partial x_2}{\partial y_1} = 0$ } kan je oplossen

→ ook hogere orde afgeleiden kunnen hieruit worden gehaald.

DATUM 140982

impliciete functie stelling

$f(x,y) = \frac{x^2}{a} + \frac{y^2}{b} - 1 = 0$

$y>0 \quad f_y = \frac{2y}{b^2} \neq 0 \qquad y^2 = b^2(1-\frac{x^2}{a^2})$

$f(x,y)=0 \Rightarrow y = \varphi(x), \; f(x,\varphi(x)) = 0$

$f(a,b)=0 \qquad f_y(a,b) \neq 0 \qquad \frac{d\varphi}{dx} = -\frac{f_x}{f_y}$

$\frac{df}{dx} = f_x + f_y\frac{d\varphi}{dx} = 0$

raaklijn $\frac{Y-y}{X-x} = \frac{d\varphi}{dx} = -\frac{\frac{2x}{a^2}}{\frac{2y}{b^2}}$

$\frac{(X-x)x}{a^2} + \frac{(Y-y)y}{b^2} = 0$

$\underbrace{f_1, f_2 \cdots\cdots f_m}_{f}$ $\quad \underbrace{x_{1,2} \cdots x_n}_{X \in U \subset \mathbb{R}^n} \underbrace{y_1, y_2 \cdots y_m}_{Y \in V \subset \mathbb{R}^m}$

$\det \frac{\partial(f_1, f_2 \cdots f_m)}{\partial(y_1, y_2 \cdots y_m)} \neq 0$ in $(a,b)$

$a \subset U$
$b \subset V$ $\qquad f(x,y) = f(a,b) \qquad y_i = \varphi_i(x)$ op $U_0$ van $a$

$f(x,\varphi(x)) = f(a,b)$

$$\frac{\partial(\varphi_1,\varphi_2\cdots\varphi_m)}{\partial(x_1,x_2\cdots x_n)} = -\left[\frac{\partial(f_1 f_2 \cdots f_m)}{\partial(y_1,y_2\cdots y_m)}\right]^{-1}\frac{\partial(f_1,f_2\cdots f_m)}{\partial(x_1,x_2\cdots x_n)}$$

voorbeeld 1. $(f_1 = F, f_2 = G)$

$f_1(x_1,x_2,y_1,y_2) = 0 \qquad f_2(x_1,x_2,y_1,y_2) = 0 \quad \to y_1 = \varphi_1(x) \quad y_2 = \varphi_2(x)$

$$\begin{bmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial x_2} \\ \frac{\partial\varphi_2}{\partial x_1} & \frac{\partial\varphi_2}{\partial x_2} \end{bmatrix} = - \quad \Delta \quad \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = -\frac{1}{\Delta}\begin{bmatrix} \det\frac{\partial(f_1,f_2)}{\partial(x_1,y_2)} & \det\frac{f(f_1,f_2)}{\partial(x_2\,y_2)} \\ \det\frac{\partial(f_1,f_2)}{\partial(y_1,x_1)} & \det\frac{\partial(f_1,f_2)}{\partial(y_1,x_2)} \end{bmatrix}$$

inverse
$$\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = -\frac{1}{\Delta}\begin{bmatrix} \frac{\partial f_2}{\partial y_2} & -\frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_1}{\partial y_1} \end{bmatrix}$$ (det Δ)

$f_1 = x_1^2 + x_2^2 + y_1^2 + y_2^2 - 14 = 0 \qquad y_1 = y_1(x_1,x_2)$

$f_2 = x_1x_2 + x_2y_1 + y_1y_2 + y_2x_1 - 8 = 0 \qquad y_2 = y_2(x_1,x_2)$

$2x_1 + 2y_1\frac{\partial y_1}{\partial x_1} + 2y\frac{\partial y_1}{\partial x_1} = 0$

$x_2 + y_2 + (x_2+y_2)\frac{\partial y_1}{\partial x_1} + (x_1+y_1)\frac{\partial y_2}{\partial x_1} = 0$

$\frac{\partial y_1}{\partial x_1} = -\frac{1}{\Delta}\det\frac{\partial(f_1,f_2)}{\partial(x_1,y_2)}$ = (20.2.

$\Delta = \det\frac{\partial(f_1 f_2)}{\partial(y_1,y_2)} = \begin{vmatrix} 2y_1 & 2y_2 \\ x_2+y_2 & x_1+y_1 \end{vmatrix} =$

$2y^2 + 2x_1y_1 - 2x_2y_2 - 2y_2$

$= \det \frac{\partial(f_1, f_2)}{\partial(x_1, y_2)} = \begin{vmatrix} 2x_1 & 2y_2 \\ x_2+y_2 & x_1+y_1 \end{vmatrix} = 2x_1^2 + 2x_1y_1 - 2y^2 - 2x_2y_2$

vraagstuk 3

$f_1 = 6 \quad f_1(x, y, v) = 0$ onafh $u$

$f_2 = H \quad f(x, u) = 0$ onafh $y, v$

$\frac{\partial f_1}{\partial x} = \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial x} = -\frac{\frac{\partial f_1}{\partial x}}{\frac{\partial f_1}{\partial v}}$

## §3 coördinatentransformaties

$\underbrace{f_1, f_2 \cdots f_n}_{f}$ van $\underbrace{x_1, x_2 \cdots x_n}_{x \subset U \subset \mathbb{R}^n}$  f een-een duidig dof  f' bestaat op f(V)

$\frac{\partial(f_1, f_2 \cdots f_n)}{\partial(x_1, x_2 \cdots x_n)} \neq 0$ op $U$

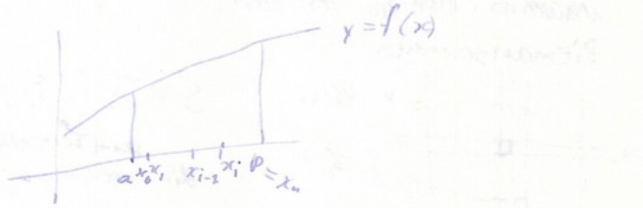
$n = 2 \quad y_1 = f_1(x_1, x_2) \quad x_1 = x_1(y_1, y_2)$

$y_2 = f_2(x_1, x_2) \quad x_2 = x_2(y_1, y_2)$

DATUM 220882

## Hoofdstuk 11 : meervoudige integralen.

$a = x_0 < x_1 \cdots < x_n = b$




$f(x)$ op $[a, b]$

$\xi_i \in [x_{i-1}, x_i] \quad \Delta x_i = x_i - x_{i-1}$

$s = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$

$\lim s = \lim_{\Delta x_i \to 0} \sum f(\xi_i)\Delta x_i = \int_a^b f(x)dx = \sigma$

$|\sigma - \sum f(\xi_i)\Delta x_i| < \varepsilon$ als verdeling fijner dan $\delta$ $\Delta x_i < \delta$

### twee voudige integraal.

$f(x, y)$ op $R$ (gesloten, begrensd gebied)

$S = \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta R_i \qquad \Delta R_i$ opp rechthoekje $R_i$ $(\xi_i, \eta_i) \in R_i$

$\lim S = \lim_{\Delta R_i \to 0} \sum f(\xi_i, \eta_i)\Delta R_i = V = \iint_R f(x, y)\, dR \to dxdy$

$|V - \sum_{\Delta R_i \to 0} \sum f(\xi_i, \eta_i)\Delta R_i| < \varepsilon$ voor $\delta > \Delta R_i$

uitrekenen

$\int_{a_1}^{a_2} f(x, y)\, dy \qquad \underbrace{x_1 \cdots x_n}_{x}$

$f(x)$ op $I$

$a \in I \quad |f(x) - f(a)| < \varepsilon$ als $|x - a| < \delta$  (opgehangen aan punt)

f uniform continuïteit op $I$ $\quad |f(x) - f(y)| < \varepsilon \quad |x - y| < \delta(\varepsilon)$  (opgehangen aan interval)

I gesloten interval.

$|f(x) - f(a)| < \varepsilon$ , $\|x - a\| < \delta(\varepsilon)$ gewoon continue

$|f(x) - f(y)| < \varepsilon \quad \|x - y\| < \delta(\varepsilon)$ uniform continue

$g(x) = \int_{x_1}^{x_2} (x,y)\, dy$

$|g(x_1) - g(x_2)| = \left| \int_{x_1}^{x_2} (f(x_1,y) - f(x_2,y)) dy \right| \le \int_{x_1}^{x_2} |f(x_1,y) - f(x_2,y)|\, dy < \varepsilon$

$|x_1 - x_2| < \delta(\varepsilon)$, $< \frac{\varepsilon}{x_2 - x_1}$

$f \to g(x)$ cont. diff^b

$\frac{dg}{dx} = \int_{\alpha_2}^{\alpha_1} f_x(x,y)\, dy$

$\alpha_1 = \varphi_1(x)$ $\quad g(x, \alpha_1, \alpha_2)$ $\quad \frac{dg}{dx} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial \alpha_1}\frac{d\alpha_1}{dx} + \frac{\partial g}{\partial \alpha_2}\frac{d\alpha_2}{dx} = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy$

$\alpha_2 = \varphi_2(x)$

$\int_{\varphi_1(x)}^{\varphi_2(x)} f_x(x,y)\, dy - f(x, \varphi(x_1)) \varphi_1' + f(x, \varphi_2(x)) \varphi_2'$

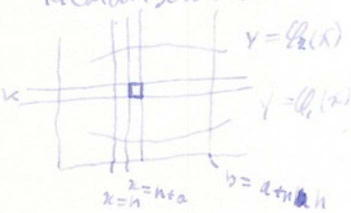

DATUM 290982

$f(x,y)$ op $R$: $a \le x \le b$, $\varphi_1(x) \le y \le \varphi_2(x)$

$\iint_R f(x,y)\, dx dy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy = \int_a^b \left\{ \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy \right\} dx = \int_a^b g(x)\, dx$

waarom kan dit zomaar?:

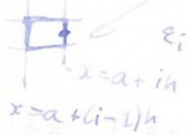
Riemansommen:




$S = \sum f(\xi_i, \eta_i)\, h \cdot k = \sum f(a + ih, \eta_i)\, hk$

punt in rechthoekje gekozen. dit kiezen we: daar.

$\left| \iint_R f(x,y)\, dx dy - \sum f(a+ih, \eta_i) hk \right| < \frac{\varepsilon}{4}$ (1)




$\xi_i = a + ih$ $\quad$ sommeren langs $x = a + ih$

$\sum f(x, y_i) k \to \int^{\varphi_2(x)}_{\varphi_1(x)} f(x,y)\, dy = g(x)$

$\sum f(x, y_i) hk \to \sum f(a+ih, \eta_i) hk \to \sum_{i=1}^{n} h\, g(a+ih)$

(k voldoende klein)

$\left| \sum f(a+ih, \eta_i) hk - \sum_{r=1}^{n} h\, g(a+ih) \right| < \frac{\varepsilon}{4}$ (2)

(1,2) $\quad \left| \iint_R f(x,y)\, dx dy - \sum h g(a+ih) \right| = \left| \iint f(x,y)\, dx dy - S \right| + \left| S - \sum g(a+ih) h \right| < \frac{\varepsilon}{2}$ (4) $\quad (< \frac{\varepsilon}{4}, < \frac{\varepsilon}{4})$

$\sum_{i=1}^{n} g(a+ih) h \to \int_a^b g(x)\, dx$ $\quad \left| \sum_{i=1}^{n} g(a+ih) h - \int_a^b g(x)\, dx \right| < \frac{\varepsilon}{2}$ (als h voldoende klein) (3)

(4,3) $\quad \left| \iint_R f(x,y)\, dx dy - \int_a^b g(x)\, dx \right| < \varepsilon$

$\iint_R f(x,y)\, dx dy = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy \right) dx$

---

$f = 1$ $\quad \iint_R dx dy = \int_a^b [\varphi_2(x) - \varphi_1(x)]\, dx = \int_a^b \varphi_2(x)\, dx - \int_a^b \varphi_1(x)\, dx$

$\iint_R (f+g)\, dx dy = \iint_R f\, dx dy + \iint_R g\, dx dy$ $\quad \iint_{R_1} f\, dx dy + \iint_{R_2} f\, dx dy = \iint_{R_1 + R_2} f\, dx dy$

de bew door terug te grijpen op Riemansommen.

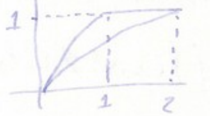
$f \ge g \to \iint f \ge \iint g$

$\left| \iint f\, dx dy \right| \le \iint |f|\, dx dy$

Hoorcollege : V. W. I  BLAD 3

$R: \sqrt{x/2} \le y \le \sqrt{x}$ op $0 \le x \le 1$  } of $0 \le y \le 1$  $y^2 \le x \le 2y^2$

$\sqrt{x/2} \le y \le 1$ op $1 \le x \le 2$



$$\iint_R \frac{\sin y}{y} = \int_1 dx \int_{2y^2} \frac{\sin y}{y} dy = \int_0^1 dx \int_{\sqrt{x/2}}^{\sqrt{x}} \frac{\sin y}{y} dy + \int_1^2 dx \int_{\sqrt{x/2}}^{1} \frac{\sin y}{y} dy$$

$$\int_0^1 dy \int_{y^2}^{2y^2} \frac{\sin y}{y} dx = \int_0^1 \frac{\sin y}{y} dy \int_{y^2}^{2y^2} dx = \int_0^1 y^2 \frac{\sin y}{y} dy = \int_0^1 y \sin y \, dy =$$ "partieel oplossen"

$R: \frac{x^2}{a} + \frac{y^2}{b} \le 1$

$$\iint dx dy = \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy = 2b\int_{-a}^{a} \sqrt{1-\frac{x^2}{a^2}} dx = 2ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi \, d\varphi = \pi ab$$

DATUM 061082

$c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$, $y = b\sqrt{1-\frac{x^2}{a}}$, $x = a$

Reductie van integralen:

$$\iint_R f(x,y)\, dx\, dy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy$$

inhoud elepsoïde: $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$

$$\iiint_R dx\, dy\, dz = \int dx \iint_{R'} dy\, dz = \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz =$$

$y = p\sin\varphi$

$$= 2c \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy = \frac{2c}{b} \int_{-a} dx \int \sqrt{\underbrace{b^2\left(1-\frac{x^2}{a^2}\right)}_{p^2} - y^2} = \frac{2c}{b} \int_{-a}^{a} dx \int_{-p}^{p} \sqrt{p^2 y^2}\, dy =$$

$$\frac{2c}{b} \int_{-a}^{a} p^2 dx \overbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi \, d\varphi}^{\frac{\pi}{2}} = \frac{\pi c}{b} \int_{-a}^{a} b^2\left(1-\frac{x^2}{a^2}\right) = \frac{\pi c}{b} \left[ b^2\left(x - \frac{x^3}{3a^2}\right)\right]_{-a}^{a} = \frac{4\pi}{3} abc$$

doorsnijdende cylinders.

$x^2 + z^2 = a^2$

$y^2 + z^2 = a^2$

$$\iiint_R dx\, dy\, dz = \int dz \iint_{R'} dy\, dx = \int_{-a}^{a} dz \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} dy \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} dx = 2\int_{-a}^{a} dz \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} \sqrt{a^2-z^2}\, dy =$$

$$4\int_{-a}^{a} (a^2-z^2)\, dz = 4\left[a^2 z - \frac{z^3}{3}\right]_{-a}^{a} = \frac{16}{3}a^3$$

~~coördinatie tr~~ <u>coördinaten transformatie</u>

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f(\varphi(t)) \frac{d\varphi}{dt} dt \qquad x = \varphi(t),\ a = \varphi(\alpha),\ b = \varphi(\beta)$$

lengte $\Delta x_i \ne \Delta t_i$ als $\xi_i = \varphi(t_i)$

$\sum f(\xi_i)\Delta x_i$  $f(\xi_i) = f[\varphi(t_i)]$

$\ne \sum f[\varphi(t_i)]\Delta t_i$

$$\sum f(\xi_i)\Delta x_i = \sum f[\varphi(\tau_i)] \frac{\Delta x_i}{\Delta t_i} \Delta t_i$$

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f[\varphi(t)] \frac{d\varphi}{dt} dt$$

meervoudige integralen.

$x = \varphi(u,v)$, $y = \psi(u,v)$

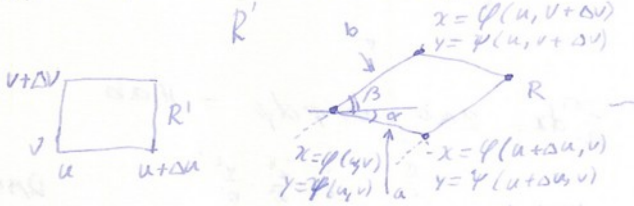
$$\iint_R f(x,y)\,dx\,dy \qquad x = \varphi(u,v) \quad y = \psi(u,v) \qquad R = R' \qquad R_i, \Delta R_i' \neq \Delta R_i$$

$$\Rightarrow \frac{\Delta R_i}{\Delta R_i'} = \left| \det \frac{\partial(\varphi,\psi)}{\partial(x,y)} \right|$$

$\downarrow$

$$\sum f(\xi_i, \eta_i)\Delta R_i = \sum f(\xi_i, \eta_i) \frac{\Delta R_i}{\Delta R_i'} \Delta R_i'$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f[\varphi(u,v), \psi(u,v)] \left| \det \frac{\partial(\varphi,\psi)}{\partial(u,v)} \right| du\,dv$$




DATUM: 131082

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f[\varphi(u,v), \psi(u,v)] \left| \det \frac{\partial \varphi, \psi}{\partial u, v} \right| du\,dv$$

$x = \varphi(u,v)$
$y = \psi(u,v)$

$$\frac{\Delta R}{\Delta R'} = \left| \det \frac{\partial(\varphi,\psi)}{\partial(u,v)} \right| \qquad \Delta R' = \Delta u \cdot \Delta v \qquad \Delta R = \begin{vmatrix} \varphi(u+\Delta u, v) - \varphi(u,v) & \varphi(u, v+\Delta v) - \varphi(u,v) \\ \psi(u+\Delta u, v) - \psi(u,v) & \psi(u, v+\Delta v) - \psi(u,v) \end{vmatrix} =$$

$$= \begin{vmatrix} a\cos\alpha & b\cos\beta \\ a\sin\beta & b\sin\beta \end{vmatrix} = ab(\cos\alpha\sin b - \cos\beta\sin\alpha) = ab\,\sin(\beta - \alpha)$$

$$\frac{\Delta R}{\Delta R'} = \begin{vmatrix} \frac{\varphi(u+\Delta u, v) - \varphi(u,v)}{\Delta u} & \frac{\varphi(u, v+\Delta v) - \varphi(u,v)}{\Delta v} \\ \frac{\psi(u+\Delta u, v) - \psi(u,v)}{\Delta u} & \frac{\psi(u, v+\Delta v) - \psi(u,v)}{\Delta v} \end{vmatrix} \xrightarrow[\lim\ \Delta v, \Delta u \to 0]{} \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix}$$

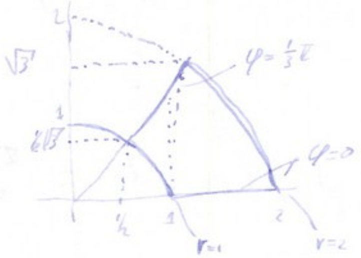
vraagstuk 1 (11-25)

$$\int_{\frac{1}{2}} \left\{ \int_{\sqrt{1-x^2}}^{x\sqrt{3}} f(x,y)\,dy \right\} dx + \int_1^2 \left\{ \int_0^{\sqrt{4-x^2}} f(x,y)\,dy \right\} dx$$

bekend

$x = r\cos\varphi$
$y = r\sin\varphi$

$$\iint_{R'} f(r\cos\varphi, r\sin\varphi)\, r\,dr\,d\varphi = \int_0^{\frac{\pi}{3}} d\varphi \int_1^2 f(r\cos\varphi, r\sin\varphi)\, r\,dr$$




vraagstuk 2

$(x^2+y^2)^2 = a^2(x^2-y^2)$ $\quad r^4 = a^2 r^2(\cos^2\varphi - \sin^2\varphi)$

$x = r\cos\varphi$
$y = r\sin\varphi$

$$r^2 = a^2 \cos(2\varphi)$$

$$\iint_R dx\,dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{a\sqrt{\cos 2\varphi}} r\,dr = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2\cos 2\varphi\,d\varphi = \frac{1}{4}a^2 \Big[\sin 2\varphi\Big]_{-\pi/4}^{\pi/4} = \frac{a^2}{2}$$

oneigenlijke integralen

$\int_a^b f(x)\,dx$, $f(x)$ in $x=a$ singulier, bv $\int_0^1 \log x\,dx$

$\llcorner \lim_{\varepsilon \downarrow 0} \int_{a+\varepsilon}^b f(x)\,dx$ — bestaat deze dan limit oneigenlijk convergent.

$$\text{Vb} \int_0^1 \ln x\,dx = \lim_{\varepsilon\downarrow 0}\int \ln x\,dx = \lim_{\varepsilon\downarrow 0} [x\ln x - x]_\varepsilon^1 = -1$$

$$\int_a^\infty f(x)\,dx = \lim_{A\to\infty}\int_a^A f(x)\,dx \qquad \text{vb} \int_1^\infty \frac{dx}{x^2} = \lim_{A\to\infty}\int_1^A \frac{dx}{x^2} = \lim_{A\to\infty}\left[-\frac{1}{x}\right]_1^A = 1$$

met meervoudige integralen onderkennen we ook twee mogelijkheden.

$\iint_R f(x,y)\,dx\,dy$ , f singulier in p

$u_k \to 0$

$R - U_k = R_k$

$= \lim_{k\to\infty} \iint_{R_k} f(x,y)\,dx\,dy$ met als extra voorwaarde, dat dit niet mag afhangen van hoe het gebiedje u er uitziet!

v.b $\lim \iint_R \ln(x^2+y^2)\,dx\,dy \to \lim_{k\to\infty} \iint r \ln r\,dr\,d\varphi = \lim_{k\to\infty} \int_0^{2\pi} d\varphi \int_{r_2(\varphi)}^{r(\varphi)} r \log r\,dr$



$\int_0^{2\pi} d\varphi \int_0^{r(\varphi)} r \log r\,dr = \int_0^{2\pi} r(\varphi)(\ln r(\varphi) - 1)\,d\varphi$ enz. (0 als r=0)

$\iint_R f(x,y)\,dx\,dy \to$ is convergent als $\lim_{n\to\infty} \iint_{R_k} f(x,y)\,dx\,dy$ bestaat.

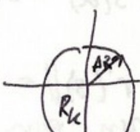
$R_1, R_2, \dots \to R$

Datum 201082

vb

$\iint_R e^{-(x^2+y^2)}\,dx\,dy$ $\quad x = r\cos\varphi$, $y = r\sin\varphi$

tweemanieren: met circels:



$A_k \to \infty$

$\Rightarrow A_k = \int_0^{2\pi}\int_0^{k} r e^{-r^2}\,dr\,d\varphi = -\frac{1}{2}\int_0^{2\pi} [e^{-r^2}]_0^k\,d\varphi = -\pi\cdot(e^{-k^2} - 1)$

$\lim_{k\to\infty} A_k = \lim_{k\to\infty} \pi(1 - e^{-k^2}) = \pi$

tweede manier.

$\iint_R e^{-(x^2+y^2)}\,dx\,dy = \lim_{k\to\infty} \int_{-A_k}^{+A_k} dx \int_{-A_k}^{+A_k} e^{-x^2} e^{-y^2}\,dy = \lim_{k\to\infty} \int_{-A_k}^{+A_k} e^{-x^2}\,dx \cdot \int_{-A_k}^{+A_k} e^{-y^2}\,dy =$

$\lim_{k\to\infty} \left\{ \int_{-A_k}^{A_k} e^{-x^2}\,dx \right\}^2 = \left\{ \int_{-\infty}^{+\infty} e^{-x^2}\,dx \right\}^2 = \pi \Rightarrow \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

Hoofdstuk 12: ruimtekrommen.

$x_1 = x_1(t)$ $\quad t \in [a,b]$

$x_2 = x_2(t)$ $\quad \underline{r}(t)$ is de vector van de oorsprong naar $(x_1, x_2, x_3)$

$x_3 = x_3(t)$

operaties op vectoren.

som $\underline{x} + \underline{y} = \underline{z}$ waarvoor geldt $z_i = x_i + y_i$

$\alpha \underline{x} = \underline{z}$ waarvoor geldt $z_i = \alpha x_i$

$\alpha(\underline{x} + \underline{y}) = \alpha\underline{x} + \alpha\underline{y}$

inwendige produkt: $(\underline{x}, \underline{y}) = \sum x_i y_i$

$\|\underline{x}\| = \sqrt{(x,x)} = \sqrt{\sum x_i^2}$

$\cos\varphi = \frac{(\underline{x},\underline{y})}{\|\underline{x}\|\cdot\|\underline{y}\|}$

$(\underline{x},\underline{y}) = \|\underline{x}\|\cdot\|\underline{y}\|\cdot\cos\varphi$

$|(\underline{x},\underline{y})| \leq \|\underline{x}\|\,\|\underline{y}\|$

uitwendig produkt:

$\underline{z} = \underline{x} \times \underline{y}$ $\quad \underline{z} \perp \underline{x}$ , $\underline{z} \perp \underline{y}$ $\quad \|\underline{z}\| = \sigma$

$\begin{cases} x_1 z_1 + x_2 z_2 + x_3 z_3 = 0 \\ y_1 z_1 + y_2 z_2 + y_3 z_3 = 0 \end{cases}$

$z_1 : z_2 : z_3 = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} : -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} : \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$

$z_1 = \rho(x_2 y_3 - x_3 y_2)$

$z_2 = \rho(x_3 y_1 - x_1 y_3)$ $\quad \rho = +1$

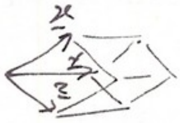
$z_3 = \rho(x_1 y_2 - x_2 y_1)$

$z_1 = x_2 y_3 - x_3 y_2$

$z_2 = x_3 y_1 - x_1 y_3$

$z_3 = x_1 y_2 - x_2 y_1$

$(\underline{x}, \underline{y} \times \underline{z}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$ $\quad (\underline{x}, \underline{y} \times \underline{z}) = (\underline{x} \times \underline{y}, \underline{z})$



$(\underline{x}, \underline{y} \times \underline{z})$ is de inhoud van het parallelepipium tussen:

~~de normale reken~~

$r(t) \rightarrow r'(t)$ nu geldt $r_i'(t) = \frac{d}{dt} r_i'(t)$

de normale reken regels gelden.

$\frac{d}{dt}(\underline{r}_1 + \underline{r}_2) = \frac{d}{dt}\underline{r}_1 + \frac{d}{dt}\underline{r}_2$

en $\frac{d}{dt}(\underline{a}, \underline{b}) =$ ~~$\frac{d}{dt}a$~~ $(\frac{d}{dt}\underline{a}, \underline{b}) + (\underline{a}, \frac{d}{dt}\underline{b})$

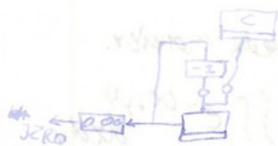
idem $\frac{d}{dt}\underline{a} \times \underline{b} = (\frac{d}{dt}\underline{a}) \times \underline{b} + \underline{a} \times (\frac{d}{dt}\underline{a})$

$\int_b^a \|r'(t)\| dt = s(a) - s(b)$ ⇓ s stijgende functie, weer nieuwe r afhankelijk van s te maken → $\|r'(s)\| = 1$

↳ lengte van een kromme.

$r(t) \begin{cases} x_1 = x_1(t) \\ x_2 = x_2(t) \\ x_3 = x_3(t) \end{cases} \quad t \in [a,b] \quad r'(t) \begin{cases} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{cases}$

$\frac{d}{dt}\|\underline{r}\| = \|\underline{r}'\|$



r differentieerd → raaklijnvector (lijk voor de hand)

$\|r'\|$ intergreert → lengte van de kromme.

$t = [a,b] \quad a = t_0 < t_1 < t_2 < \dots t_{i-1} < t_i < \dots < t_n = b$

$P_{i-1} \quad P_i$

$\int_a^b \|\underline{r}'\| dt \qquad s(t) = \int_a^t \|r'(t)\| dt$

$s(t)$ stijgend, $t = t(s) \rightarrow \begin{matrix} x_1(s) \\ x_2(s) \\ x_3(s) \end{matrix} \quad \|r'(s)\| = 1 \qquad \frac{dx_i}{dt} = \frac{dx_i}{ds}\frac{ds}{dt} = \frac{dx_i}{ds}$

$\sum\left(\frac{dx}{dt}\right)^2 = \|r'(t)\| \sum\left(\frac{dx_i}{ds}\right)^2 \rightarrow \sum\left(\frac{dx_i}{ds}\right)^2 = 1$

$\uparrow \sum\left(\frac{dx_i}{dt}\right)^2$

vb $x_1 = r\cos t \quad x_2 = r\sin t \quad x_3 = ct$

$s(t) = \int_0^t \sqrt{r^2\sin^2 t + r^2\cos^2 t + c^2}\, dt = \int_0^t \sqrt{r^2 + c^2}\, dt = s\sqrt{r^2 + c^2}$

$f(x_1, x_2, x_3) = f(\underline{r}) \cdot \underline{r} \begin{cases} x_1 \\ x_2 \\ x_3 \end{cases} \quad \frac{\partial f}{\partial x_1} = \lim_{h \to 0} \frac{f(x_1 + h, x_2, x_3) - f(x_1, x_2, x_3)}{h} = \lim_{h \to 0} \frac{f(\underline{r} + h\underline{e}_1) - f(\underline{r})}{h}$

$x \in \mathbb{R}$

$f'(\underline{r}; \underline{a}) = \lim_{h \to 0} \frac{f(\underline{r} + h\underline{a}) - f(\underline{r})}{h}$

$f(\underline{r} + h\underline{a}) - f(\underline{r}) = f(x_1 + ha_1, x_2 + ha_2, x_3 + ha_3) - f(x_1, x_2, x_3) =$

$= f_{x_1} h a_1 + f_{x_2} h a_2 + f_{x_3} h a_3 + \varepsilon_1 h a_1 + \varepsilon_2 h a_2 + \varepsilon_3 h a_3$ eerste orde Taylor ontw.

($\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, $\varepsilon_3 \to 0$)

$f'(\underline{r};\underline{a}) = f_{x_1} a_1 + f_{x_2} a_2 + f_{x_3} a_3 = (\text{Grad } f, \underline{a})$

$\text{grad} f \begin{cases} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_i} \end{cases}$

$(\text{grad } f, \underline{a}) = \|\text{grad} f\| \|\underline{a}\| \cos\varphi = \|\text{grad} f\| \cos\varphi$

lengte $=1$

$(\text{grad } f, \underline{a}) = \|\text{grad} f\|$ als $\varphi = 0$ grad f wijst dus in de richting waarin f het sterkst verandert.

$\left.\begin{matrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{matrix}\right\} \subset \quad \frac{df[x(t)]}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}\frac{dx_3}{dt} = (\text{grad } f, \underline{r}') = f'(\underline{r}; \underline{r}')$

$f(x_1, x_2, x_3) = \text{const}$ (Niveau-Opp) $\subset$ op dit opp $(\text{grad } f, \underline{r}') = 0$

grad f $\perp$ op het nivo oppervlak.